Analytic expression for exact-ground-state energy based on an operator method for a class of anharmonic potentials

L. C. Kwek, Yong Liu, C. H. Oh,* and Xiang-Bin Wang
Physics Department, National University of Singapore, Kent Ridge, Singapore, 119260
(Received 10 May 2000; published 13 October 2000)

A general procedure based on shift operators is formulated to deal with anharmonic potentials. It is possible to extract the ground-state energy analytically using our method provided certain consistency relations are satisfied. Analytic expressions for the exact-ground-state energy have also been derived specifically for a large class of the one-dimensional oscillator with cubic-quartic anharmonic terms. Our analytical results can be used to check the accuracy of existing numerical methods, for instance the method of state-dependent diagonalization. Our results also agree with the asymptotic behavior in the divergent pertubative expansion of the quartic harmonic oscillator.

PACS number(s): 03.65.Fd, 34.20.Gj

I. INTRODUCTION

The operator method has been widely used as an elegant analytical tool in quantum mechanics [1,2] and quantum statistics [3] for studying exactly solvable models. Since the publication of a remarkable paper by Delbecq and Quesne [4], the operator method has been extended to many other possible physical applications and realizations [5–13]. Indeed, the generalization of the notion of ''shift operators'' or ''ladder operators'' as a spectrum generating algebra can be effectively studied both mathematically and physically in terms of ''nonlinear algebra'' introduced by Delbecq and Quesne [4]. In fact, Chen *et al.* [12] has recently reproduced the results of the Bethe Ansatz [14–16] for the *XXX* model through the shift operator method. A general observation in these works appears to be a possibility of treating the well-known technique of the algebraic Bethe Ansatz for the quantum many-body problem as a special case of the shift operator method. Thus, all previous works evaluated using the algebraic Bethe Ansatz should hopefully be possible using a nonlinear algebraic method via shift operators. A further implication of this notion is the emergence of an underlying Yangian algebra. Furthermore, the factorization method employed in supersymmetric quantum mechanics (SQM) [17] can also be deemed as a special case of the nonlinear Delbecq-Quesne algebra.

In many realizations of physical models, the operator method can provide us with an alternative and a clearer picture regarding the analytical determination of energy eigenstates and eigenvalues. Specifically, it has been found to be a useful tool in the solution of bound-state problems such as the string postulate in the Bethe Ansatz or the raising and lowering of energy levels in SQM. As in the Bethe Ansatz and SQM, knowledge of a certain reference state, such as the highest weight state in the Bethe Ansatz or the ground-state ansatz for the superpartner potential in SQM, in the operator method [1] permits the analysis and determination of the full spectrum of a physical system. Moreover, as mentioned in Ref. [2], the knowledge of the ground-state wave function also determines the potential and hence the Hamiltonian of a system up to a constant value. However, in principle, given the Hamiltonian of a physical system, surely it should be possible to determine all the energy levels. In this paper we show that, provided certain consistency relations are satisfied, the entire spectrum of a given Hamiltonian can be determined by the shift operator method even if there is no prior knowledge of a reference state.

To illustrate the power of the shift operator method, we apply the technique to a one-dimensional oscillator with anharmonic potentials. The study of the anharmonic potential [18–20] has always been an exciting and interesting field due to its broad applications in quantum field theory [21,22], nuclear models, atomic and molecular physics [23–25], condensed matter physics [26,27], statistical physics, and chemical physics [28]. Indeed, numerous numerical methods, including renormalized strong-coupling expansion, perturbation expansion, supersymmetric quantum mechanics, WKB, iteration based on the generalized Bloch equation, state-dependent diagonalization, the Hill determinant method, the phase-integral approach, iterative Bogoliubov transformations, the eigenvalue moment method, the perturbative variation, and the algebraic method, have been proposed to investigate these anharmonic potentials [29–45]. However, due to its inherent intractability, no analytic solution for the energy spectrum and eigenstates has been obtained so far except in some special cases. Indeed, we have partially resolved some of these technical issues and obtained analytic solutions for a wide class of anharmonic potentials under certain conditions.

It is instructive to see how SQM can be regarded as a special case of the nonlinear algebraic operator method. Before we proceed further to show this point, we first recall some relevant definitions in the operator method and nonlinear algebra [1,4,11,13]. Let $\hat{H}$ be an observable satisfying the eigenfunction equation $\hat{H}|\psi\rangle = E|\psi\rangle$. The operators $L^{\pm}$ satisfying

$$[\hat{H}, L^{\pm}] = L^{\pm} f^{\pm}(\hat{H}), \tag{1}$$

*Email address: phyohch@nus.edu.sg

are called ''shift operators'' of $\hat{H}$. In Eq. (1), $f^{\pm}(\hat{H})$ are real functions of $\hat{H}$ so that the values of its action on eigenstates of H are reals. Thus, these values can be interpreted as energy gaps if $\hat{H}$ is the Hamiltonian of the system. Nevertheless, it is not necessary for the mutually adjoint condition

$$(L^+)^\dagger = L^-$$

to hold and there is generally no constraint on the commutation relation between L^+ and L^-, i.e., $[L^+,L^-]$. However, if the mutually adjoint condition is satisfied, we obtained the ''nonlinear algebra'' [4] defined by the relations

$$[\hat{H},L^+] = f(\hat{H})L^+,$$

$$[\hat{H},L^-] = -L^- f(\hat{H}),$$

$$[L^-,L^+] = g(\hat{H}).$$

A simple rearrangement of the above relations gives

$$[\hat{H} - f(\hat{H})]L^+ = L^+\hat{H},$$

$$L^-[\hat{H} - f(\hat{H})] = \hat{H}L^-. \tag{2}$$

It is interesting to compare Eq. (2) with the analogous relations in SQM [17]

$$\hat{H}_1 Q^+ = Q^+ \hat{H}_2,$$

$$Q^- \hat{H}_1 = \hat{H}_2 Q^-,$$

$$Q^- = (Q^+)^\dagger, \tag{3}$$

and identify

$$Q^+ \sim L^+, \quad Q^- \sim L^-,$$

$$\hat{H}_1 \sim \hat{H} - f(\hat{H}), \quad \hat{H}_2 \sim \hat{H}. \tag{4}$$

Hence, one can always regard SQM as a specific type of nonlinear algebraic method.

The purpose of this paper is to describe the operator method [1,11–13] and apply it to study the one-dimensional oscillator with anharmonic potentials. In Sec. II we show how a shift operator method can be formulated using a simple instructive example of the harmonic oscillator. The general procedures to solve anharmonic potentials are presented in Sec. III. In Sec. IV we solve the relevant equations for the one-dimensional oscillator with cubic-quartic potential and show that we can get the exact analytical expression for the ground-state energy provided certain consistency relations are satisfied. We then compare our analytical result with some numerical results in Sec. V. Our analytical expressions can be used to verify the accuracy of existing numerical results. We conclude with some brief remarks in Sec. VI.

II. APPLYING THE OPERATOR METHOD TO THE HARMONIC OSCILLATOR

In this section we apply the operator method to the harmonic oscillator. The Hamiltonian of the harmonic oscillator is given by

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2). \tag{5}$$

Together with the well-known commutation relation between $\hat{x}$ and $\hat{p}$, i.e., $[\hat{x},\hat{p}] = i$ $(\hbar = 1)$, we get

$$[\hat{H},\hat{x}] = -i\hat{p},$$

$$[\hat{H},\hat{p}] = i\hat{x}. \tag{6}$$

Using the notation

$$[\hat{H},(\hat{x},\hat{p})] := ([\hat{H},\hat{x}],[\hat{H},\hat{p}]),$$

Eq. (6) can also be rewritten as

$$[\hat{H},(\hat{x},\hat{p})] = (\hat{x},\hat{p})\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{7}$$

$$\equiv (\hat{x},\hat{p})M \tag{8}$$

where the matrix M is called the ''coefficient matrix.'' To diagonalize the coefficient matrix, we look for a transformation U such that $M = UDU^{-1}$, where D is a diagonal matrix. In general, provided all the entries of U commute with the Hamiltonian $\hat{H}$, the entries of the matrix U are functions of c numbers, the Hamiltonian or any conserved quantity $\hat{I}$ of the system. Thus, we have

$$\begin{aligned}
&[\hat{H},(\hat{x},\hat{p})U(c,\hat{H},\hat{I})] \\
&= (\hat{x},\hat{p})U(c,\hat{H},\hat{I})U^{-1}(c,\hat{H},\hat{I})\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}U(c,\hat{H},\hat{I}) \\
&= (\hat{x},\hat{p})U(c,\hat{H},\hat{I})\begin{pmatrix} \lambda_1(c,\hat{H},\hat{I}) & 0 \\ 0 & \lambda_2(c,\hat{H},\hat{I}) \end{pmatrix}.
\end{aligned} \tag{9}$$

It is easy to see that

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix},$$

so that Eq. (7) can be rewritten as

$$\left[\hat{H},(\hat{x},\hat{p})\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\right] = (\hat{x},\hat{p})\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

Identifying

$$(\hat{a}^+,\hat{a})=(\hat{x},\hat{p})\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix},$$

we obtain

$$[\hat{H},\hat{a}^+]=\hat{a}^+,$$

$$[\hat{H},\hat{a}]=-\hat{a}.$$

The ''creation'' and the ''annihilation'' operators are thus obtained naturally. Following the literature, we can also call them ''shift'' or ''ladder'' operators. Although the application of the shift operator to harmonic oscillator is very simple, it can still serve an illustrative example.

For the harmonic oscillator, a closed algebra is obtained. This closure in the algebra enables the whole spectrum of the system to be determined using only one pair of shift operators, in which one of the operators raises while the other lowers the energy levels. In general, the existence of a closed algebra is not assured, and so we usually have to deal with an unclosed algebra with infinitely many shift operators. Thus, we have to use infinitely many shift operators pairs to generate the whole spectrum. Moreover, corresponding to each energy level, we have a pair of distinct shift operators in which one raises while the other lowers the energy level.

It is interesting to note that the eigenvalues of the coefficient matrix correspond to the amount of shifted energy associated with the various shift operators. And, except for some special Hamiltonians, the coefficient matrix, its eigenvalues, and the transformation matrix, are all functions of the Hamiltonian. In the nonlinear algebraic method, the energy shifts are no longer uniform and the gap between any two adjacent energy levels is related to their relative positions in the spectrum. Thus, we find that the shift operator method can provide us with a clearer physical picture for the mapping of the energy level in a physical system.

Note that we can only extract the energy gaps rather than the energy levels. Moreover, we have no information regarding to the energy of the ground state [1]. However, in some cases, it is possible to determine the exact energy of the ground state. This last result constitutes the gist of one of the most important aspects of our paper.

III. OPERATOR METHOD TO ANHARMONIC OSCILLATOR: THE GENERAL PROCEDURE

In this section we confine ourselves to the one-dimensional harmonic oscillator in which the anharmonic potential contains cubic and quartic terms. Nevertheless, the procedure developed here is very general and can be applied to other types of anharmonic potentials. The Hamiltonian is given by

$$\hat{H}=-\frac{d^2}{dx^2}+\alpha\hat{x}^2+\beta\hat{x}^3+\gamma\hat{x}^4, \tag{11}$$

and it is easy to show that

$$[\hat{H},\hat{x}^n]=-2n\hat{x}^{n-1}\frac{d}{dx}-n(n-1)\hat{x}^{n-2}, \tag{12}$$

$$\begin{aligned}\left[\hat{H},\hat{x}^n\frac{d}{dx}\right]=&-n(n-1)\hat{x}^{n-2}\frac{d}{dx}+2n\hat{x}^{n-1}\hat{H}\\ &-2\alpha(n+1)\hat{x}^{n+1}-2\beta(n+\tfrac{3}{2})\hat{x}^{n+2}\\ &-2\gamma(n+2)\hat{x}^{n+3}.\end{aligned} \tag{13}$$

Rewriting Eqs. (12) and (13) into matrix form, we have

$$\begin{aligned}&[\hat{H},(\hat{x},\hat{x}^2,\hat{x}^3,\ldots,\hat{x}^n,\ldots)]\\ &\quad=(\hat{x},\hat{x}^2,\hat{x}^3,\ldots,\hat{x}^n,\ldots)M_1\\ &\qquad+\left(\frac{d}{dx},x\frac{d}{dx},x^2\frac{d}{dx},\ldots,x^n\frac{d}{dx},\ldots\right)N_1+L_1,\end{aligned} \tag{14}$$

$$\begin{aligned}&\left[\hat{H},\left(\frac{d}{dx},x\frac{d}{dx},x^2\frac{d}{dx},\ldots,x^n\frac{d}{dx},\ldots\right)\right]\\ &\quad=(\hat{x},\hat{x}^2,\hat{x}^3,\ldots,\hat{x}^n,\ldots)M_2\\ &\qquad+\left(\frac{d}{dx},x\frac{d}{dx},x^2\frac{d}{dx},\ldots,x^n\frac{d}{dx},\ldots\right)N_2+L_2,\end{aligned} \tag{15}$$

where

$$M_1=\begin{pmatrix}0&0&-6&0&0&\cdots&\cdots&\cdots\\ 0&0&0&-12&0&\cdots&\cdots&\cdots\\ 0&0&0&0&-20&\cdots&\cdots&\cdots\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&0&0&0&\cdots&-n(n-1)&\cdots\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}, \tag{16}$$

$$N_1=\begin{pmatrix} -2 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & -4 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & -6 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{17}$$

$$L_1=(0,-2,0,0,\ldots,0,\ldots), \tag{18}$$

$$M_2=\begin{pmatrix} -2\alpha & 0 & 4\hat{H} & 0 & 0 & & \cdots & \cdots \\ -3\beta & -4\alpha & 0 & 4\hat{H} & 0 & & \cdots & \cdots \\ -4\gamma & -5\beta & -6\alpha & 0 & 8\hat{H} & & \cdots & \cdots \\ 0 & -6\gamma & -7\beta & -8\alpha & 0 & \ddots & \cdots & \cdots \\ 0 & 0 & -8\gamma & -9\beta & -10\alpha & \ddots & 2n\hat{H} & \cdots \\ 0 & 0 & 0 & -10\gamma & -11\beta & \ddots & 0 & \cdots \\ 0 & 0 & 0 & 0 & -12\gamma & \ddots & -2(n+1)\alpha & \cdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -2\left(n+\frac{3}{2}\right)\beta & \cdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -2(n+2)\gamma & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \end{pmatrix}, \tag{19}$$

$$N_2=\begin{pmatrix} 0 & 0 & -2 & 0 & 0 & & \cdots & \cdots \\ 0 & 0 & 0 & -6 & 0 & & \cdots & \cdots \\ 0 & 0 & 0 & 0 & -12 & & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -n(n+1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \end{pmatrix}, \tag{20}$$

and

$$L_2=(0,2\hat{H},0,0,\ldots,0,\ldots). \tag{21}$$

It is interesting to note that for other types of potentials, all matrices except for M_2 are exactly the same.

From Eqs. (14) and (15), we find

$$\left[\hat{H},(\hat{x},\hat{x}^2,\hat{x}^3,\ldots,\hat{x}^n,\ldots)R+\left(\frac{d}{dx},x\frac{d}{dx},x^2\frac{d}{dx},\ldots,x^n\frac{d}{dx},\ldots\right)S\right]$$
$$=(\hat{x},\hat{x}^2,\hat{x}^3,\ldots,\hat{x}^n,\ldots)(M_1R+M_2S)+\left(\frac{d}{dx},x\frac{d}{dx},x^2\frac{d}{dx},\ldots,x^n\frac{d}{dx},\ldots\right)(N_1R+N_2S)+(L_1R+L_2S). \tag{22}$$

Thus, if the following relations exist:

$$M_1R+M_2S=RT, \tag{23}$$

$$N_1R+N_2S=ST, \tag{24}$$

then we have

$$\left[\hat{H},(\hat{x},\hat{x}^2,\hat{x}^3,\ldots,\hat{x}^n,\ldots)R+\left(\frac{d}{dx},x\frac{d}{dx},x^2\frac{d}{dx},\ldots,x^n\frac{d}{dx},\ldots\right)S\right]$$
$$=\left\{(\hat{x},\hat{x}^2,\hat{x}^3,\ldots,\hat{x}^n,\ldots)R\right.$$

$$+\left(\frac{d}{dx},x\frac{d}{dx},x^2\frac{d}{dx},\ldots,x^n\frac{d}{dx},\ldots\right)S\Big\}T$$
$$+(L_1R+L_2S). \tag{25}$$

To diagonalize the coefficient matrix T, let us suppose that U is the transformation matrix needed. The eigenvalues and the corresponding shift operators can therefore be written as

$$\Lambda=U^{-1}TU=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n,\ldots) \tag{26}$$

and

$$(\hat{A}_1,\hat{A}_2,\hat{A}_3,\ldots,\hat{A}_n,\ldots)$$
$$=\Big\{(\hat{x},\hat{x}^2,\hat{x}^3,\ldots,\hat{x}^n,\ldots)R$$
$$+\left(\frac{d}{dx},x\frac{d}{dx},x^2\frac{d}{dx},\ldots,x^n\frac{d}{dx},\ldots\right)S\Big\}U$$
$$+(L_1R+L_2S)U\Lambda^{-1}, \tag{27}$$

respectively. In fact, Eq. (25) can be written into a more succinct form as

$$[\hat{H},(\hat{A}_1,\hat{A}_2,\hat{A}_3,\ldots,\hat{A}_n,\ldots)]$$
$$=(\hat{A}_1,\hat{A}_2,\hat{A}_3,\ldots,\hat{A}_n,\ldots)$$
$$\times\begin{pmatrix}\lambda_1&0&0&&\cdots&\cdots\\0&\lambda_2&0&&\cdots&\cdots\\0&0&\lambda_3&&\cdots&\cdots\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&\lambda_n&\cdots\\\vdots&\vdots&\vdots&&\vdots&\vdots\end{pmatrix}, \tag{28}$$

or

$$[\hat{H},\hat{A}_1]=\hat{A}_1\lambda_1,$$
$$[\hat{H},\hat{A}_2]=\hat{A}_2\lambda_2,$$
$$\cdots,$$
$$[\hat{H},\hat{A}_n]=\hat{A}_n\lambda_n,$$
$$\cdots.$$

Thus, in general we get an unclosed algebra with infinitely many shift operators. In case of the usual harmonic oscillator, we have a pair of shift operators that can generate the entire spectrum. In the present case, we have infinitely many pairs of shift operators in which each pair is responsible for raising and lowering the corresponding energy level. The whole spectrum can only be generated by the infinite set of the shift operators acting on the ground state. Hence, it is natural to see that all the eigenvalues of the coefficient matrix are only dependent on the energy of the ground state. That is, λ_i, $(i=1,2,3,\ldots,n,\ldots)$ are the functions of the ground-state energy. Therefore, we can identify the operator $\hat{H}$ in the coefficient matrix T as the ground-state energy. Based on this observation, we can then get the analytic expression for the energy of the ground state provided certain consistency relations hold.

It remains to solve for the matrices R, S, and T given the matrices M_1, M_2, N_1, and N_2. Once we have obtained the matrices R,S,T and diagonalize T, we effectively obtained all the shift operators and we can then reconstruct the full spectrum. It is instructive to note that one of the matrices R or S is redundant and we set it to unity. For convenience, let us set S in Eqs. (23) and (24) to unity so that the problem is reduce to the solution of R,T from the following equations:

$$M_1R+M_2=RT, \tag{29}$$

$$N_1R+N_2=T, \tag{30}$$

and the diagonalization of T. Note that although we have considered a specific form for the potential, the procedure developed here is very general and can be applied to more complicated cases.

IV. ANALYTIC EXPRESSION FOR THE GROUND-STATE ENERGY: A CLASS OF ANHARMONIC POTENTIALS

To solve Eqs. (29) and (30), we first make the following observation. If we define

$$G=\begin{pmatrix}1&0&0&\cdots&\cdots&\cdots\\0&2&0&\cdots&\cdots&\cdots\\0&0&3&\cdots&\cdots&\cdots\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&n&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}, \tag{31}$$

$$P=\begin{pmatrix}0&1&0&0&\cdots&\cdots&\cdots\\0&0&1&0&\cdots&\cdots&\cdots\\0&0&0&1&\cdots&\cdots&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&0&\cdots&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix},$$

$$Q=\begin{pmatrix}0&0&0&\cdots&\cdots&\cdots\\1&0&0&\cdots&\cdots&\cdots\\0&1&0&\cdots&\cdots&\cdots\\0&0&1&\cdots&\cdots&\cdots\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}, \tag{32}$$

we have

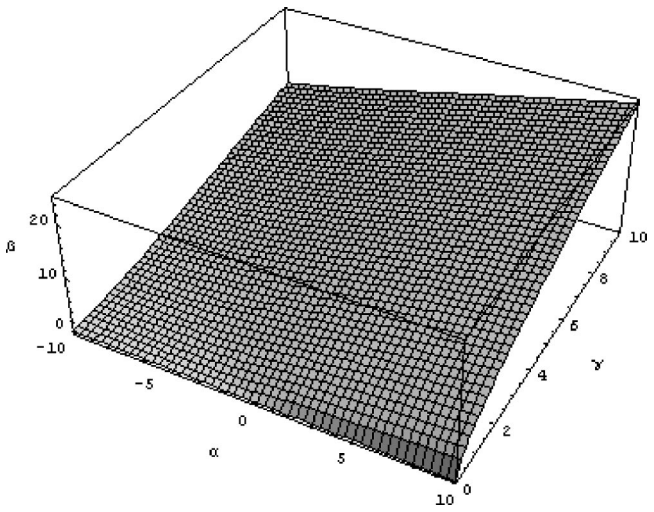

FIG. 1. Dependence of β on γ and α.

$$M_1 = -PGPG,$$
$$M_2 = -2\alpha G + 2\hat{H}PGP - 2\gamma QGQ - \beta(2G-1)Q,$$
$$N_1 = -2G,$$
$$N_2 = -GPGP. \tag{33}$$

Here, the matrices G, P, and Q are nothing but representations of particle numbers and creation and annihilation operators for the usual harmonic oscillator in Fock space. It is easy to check that

$$[G,P] = -P, \quad [G,Q] = Q, \quad PQ = 1,$$
$$QP = \mathrm{diag}(0,1,1,\ldots,1,\ldots). \tag{34}$$

Furthermore, if we set

$$W = N_1 R,$$

then Eq. (30) becomes

$$T = W + N_2 \quad R = N_1^{-1} W.$$

From Eq. (33) we notice that $N_1 M_1 N_1^{-1} = N_2$, and as a consequence, we can combine Eq. (29) and the above expression for the matrix T, to yield the expression

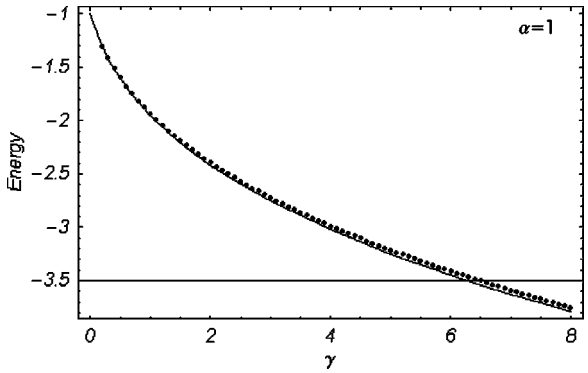

FIG. 2. Comparison of our analytic result (bold line) with that obtained by the state-dependent diagonalization method (represented by the data points) for $\alpha = 1$.

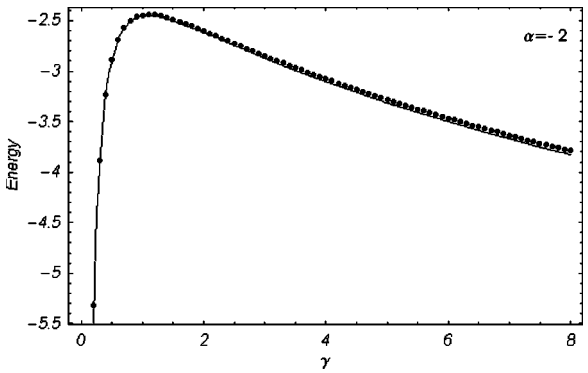

FIG. 3. Comparison of our analytic result (bold line) with that obtained by the state-dependent diagonalization method (represented by data points) for $\alpha = -2$.

$$W^2 + [W,N_2] - N_1 M_2 = 0. \tag{35}$$

Unfortunately, due to the infinite dimensionality of the matrices, the solution, except in some special cases, is not known.

In the case of the one-dimensional oscillator with the cubic-quartic term, if we let W assume the following form:

$$W = 2(aG + bGQ + cGP),$$

it is not difficult to show using Eq. (34) that

$$W^2 + [W,N_2] - N_1 M_2 = 4\{(a^2 + 2bc - \alpha)G^2 + (ab - \tfrac{1}{2}\beta) \times (2G^2 - G)Q + (b^2 - \gamma) \times GQGQ(ac + b)(2G^2 + G)P + (c^2 + a + \hat{H})GPGP\}. \tag{36}$$

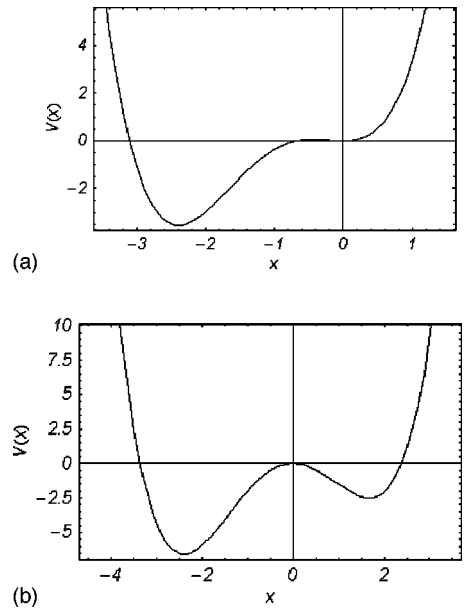

FIG. 4. The potential $V(x)$ as a function of x for (a) $\alpha = 1, \gamma = 1/2$ and (b) $\alpha = -2, \gamma = 1/4$.

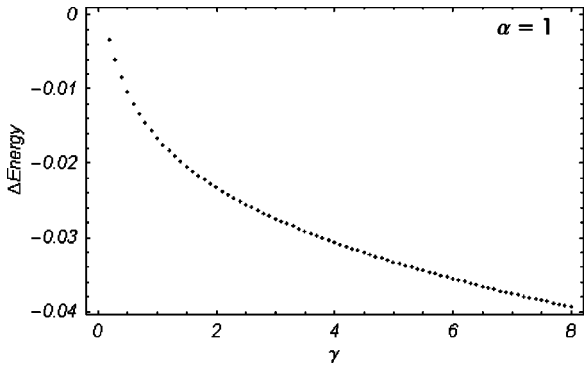

FIG. 5. Difference in the energy between the analytic and the numerical data as a function of γ for $\alpha=1$.

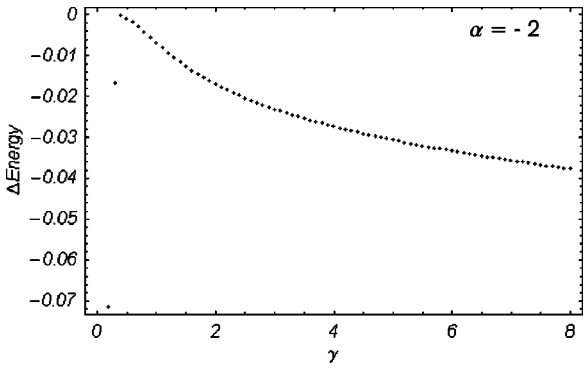

FIG. 6. Difference in the energy between the analytic and the numerical data as a function of γ for $\alpha=-2$.

Therefore, consistency naturally requires the following conditions:

$$a^2+2bc-\alpha=0, \tag{37}$$

$$ab-\frac{1}{2}\beta=0, \tag{38}$$

$$b^2-\gamma=0, \tag{39}$$

$$ac+b=0, \tag{40}$$

$$c^2+a+\hat{H}=0, \tag{41}$$

in order that Eq. (35) be satisfied. In particular, Eq. (41) gives the ground-state energy of the system.

As explained in the preceding section, we have equated $\hat{H}$ to one of its eigenvalues, namely the ground-state energy. Thus, in order that the full spectrum be generated from the infinitely many raising operators acting on the ground state, so that each raising operator generates its own corresponding energy level via its action on the ground state, the consistency relations in Eqs. (37)–(41) must hold.

It is easy to see that the solution to Eqs. (37)–(41) is

$$a=\frac{\beta}{2\sqrt{\gamma}}, \quad b=\sqrt{\gamma}, \quad c=-2\frac{\gamma}{\beta}, \tag{42}$$

$$\hat{H}=-4\frac{\gamma^2}{\beta^2}-\frac{\beta}{2\sqrt{\gamma}}, \tag{43}$$

$$\frac{\beta^2}{4\gamma}-4\frac{\gamma^{3/2}}{\beta}-\alpha=0, \tag{44}$$

and

$$a=-\frac{\beta}{2\sqrt{\gamma}}, \quad b=-\sqrt{\gamma}, \quad c=-2\frac{\gamma}{\beta}, \tag{45}$$

$$\hat{H}=-4\frac{\gamma^2}{\beta^2}+\frac{\beta}{2\sqrt{\gamma}}, \tag{46}$$

$$\frac{\beta^2}{4\gamma}+4\frac{\gamma^{3/2}}{\beta}-\alpha=0. \tag{47}$$

The additional constraint Eq. (44) [or Eq. (47)] seems to indicate that the ansatz for W may be too simplistic. However, we do not have a direct solution of W from Eq. (35) for the given N_1, N_2, and M_2. Despite all these, we can still obtain an analytic result for the exact energy of the ground state for the given anharmonic potential and this result can prove to be valuable for analyzing the accuracy of existing numerical methods. Besides, as mentioned before, it is anticipated that the operator method can provide us with an analytic tool for investigating the ground state [1], something which is not possible using existing numerical approaches.

To compute the energy levels of the excited states, we need to diagonalize the following infinite dimensional matrix:

$$T=\begin{pmatrix} 2a & 2c & -2 & 0 & 0 & & \cdots & \cdots \\ 4b & 4a & 4c & -6 & 0 & & \cdots & \cdots \\ 0 & 6b & 6a & 6c & -12 & & \cdots & \cdots \\ 0 & 0 & 8b & 8a & 8c & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & 10b & 10a & \ddots & -n(n+1) & \cdots \\ 0 & 0 & 0 & 0 & 12b & \ddots & 2nc & \cdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 2na & \cdots \\ 0 & 0 & 0 & 0 & 0 & & 2nb & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \end{pmatrix}$$

with a, b, and c given by Eqs. (42) or (45). Once we have diagonalized this matrix, using the result of the ground-state energy, all the other energy levels can be obtained. Unfortunately, we do not know any method for diagonalizing it at the present moment despite the apparent simplicity and symmetry in the matrix.

V. COMPARISON OF NUMERICAL RESULTS WITH THE ANALYTIC RESULTS FOR GROUND-STATE ENERGY

In this section we use our analytic results to check against previous numerical computations. For numerical simulation we use the method proposed by Ho *et al.* [30] using the state-dependent diagonalization method. As claimed in Ref. [30], this method is very accurate and efficient compared to other numerical methods for calculating the energy eigenvalues and eigenfunctions of the one-dimensional harmonic oscillator with anharmonic potentials.

Due to the consistency requirements in Eqs. (44) or (47), the three coefficients α, β, and γ in

$$V(x)=\alpha x^2+\beta x^3+\gamma x^4$$

are not mutually independent. It is not difficult to solve for β using Eqs. (44) or (47) for given values of α and γ. The final solution is

$$\beta=2\sqrt{\gamma}\left[\frac{\alpha/3}{(\gamma+\sqrt{\gamma^2-(\alpha/3)^3})^{1/3}}+(\gamma+\sqrt{\gamma^2-(\alpha/3)^3})^{1/3}\right] \tag{48}$$

corresponding to Eq. (44) and $-\beta$ corresponding to Eq. (47), with the ground-state energy being the same in both cases. Specifically, the ground-state energy is given by

$$E_0=-4\frac{\gamma^2}{\beta^2}-\frac{\beta}{2\sqrt{\gamma}}. \tag{49}$$

The dependence of β on α and γ is shown in Fig. 1.

Setting $\alpha=1,-2$, the values of E_0 as a function of γ for the analytic formula and the numerical simulation using the state-dependent diagonalization [30] are shown, respectively, in Figs. 2 and 3 with the corresponding potential for particular values of γ as a function of x shown in Fig. 4. The difference in the energy computed from the numerical simulation and the analytic formula as a function of γ is also plotted in Figs. 5 and 6 for $\alpha=1$ and $\alpha=-2$, respectively. As shown in the figures, the numerical simulation using state-dependent diagonalization is in excellent agreement with our analytical formula. Moreover, we see that the state-dependent-diagonalization method provides sufficiently high accuracy for all practical purposes.

VI. CONCLUSIONS

In summary, we have formulated a general procedure for extracting the full spectrum of a physical system with arbitrary potentials using the operator method. We have applied it to solve the one-dimensional harmonic oscillator with an anharmonic problem. The analytic expression for the ground-state energy is obtained for a large class of anharmonic potentials. Our results can be used to verify the accuracy of existing numerical methods. However, in order to get the full spectrum, we need to diagonalize an infinite-dimensional matrix. This last problem is, to our knowledge, an open mathematical question except for some very special cases.

Our method has also confirmed the notion that once the Hamiltonian of certain system is given, all the energy levels can be determined through the operator method even without any prior knowledge of a reference state, provided certain consistency relations hold, a feat considered impossible hitherto.

Finally, from Eqs. (48) and (49), it easy to see that in the limit when $\gamma\to\infty, E_0\to-\frac{3}{2}(2\gamma)^{1/3}$, confirming an earlier analysis on the mathematical properties of ground-state energy obtained in Refs. [46,36].

ACKNOWLEDGMENTS

This work has been supported by NUS Research Grant No. RP3982713. We would also like to thank Professor A. Turbiner (UNAM) and Professor C. Schwartz (Berkeley) for interesting discussions.

[1] O.L. De Lange and R.E. Raab, *Operator Methods in Quantum Mechanics* (Clarendon Press, Oxford, 1991).
[2] I.L. Cooper, J. Phys. A **26**, 1601 (1993).
[3] C.N. Yang, Phys. Rev. Lett. **63**, 2144 (1989).
[4] C. Delbecq and C. Quesne, J. Phys. A **26**, L127 (1993).
[5] D.J. Fernandez, M.L. Glasser, and L.M. Nieto, Phys. Lett. A **240**, 15 (1998).
[6] A.B. Balantekin, Phys. Rev. A **57**, 4188 (1998).
[7] A. Gangopadhyaya, J.V. Mallow, and U.P. Sukhatme, Phys. Rev. A **58**, 4287 (1998).
[8] J.I. Diaz, J. Negro, L.M. Nieto, and O. Rosas-Ortiz, J. Phys. A **32**, 8447 (1999).
[9] R. Dutt, A. Gangopadhyaya, C. Rasinariu, and U. Sukhatme, Phys. Rev. A **60**, 3482 (1999).
[10] A. Andrianov, F. Cannata, M. Ioffe, and D. Nishnianidze, e-print quant-ph/990205.
[11] Jing-Ling Chen, Yong Liu, and Mo-Lin Ge, J. Phys. A **31**, 6473 (1998); Jing-Ling Chen, Yong Liu, and Yuan-Chao Zhao, Commun. Theor. Phys. **32**, 253 (1999); C. Quesne, J. Phys. A **32**, 6705 (1999).
[12] Jing-Ling Chen, Mo-Lin Ge, and Kang Xue, Phys. Rev. E **60**, 1486 (1999).
[13] M.L. Ge, L.C. Kwek, Yong Liu, C.H. Oh, and Xiang-Bin Wang, e-print quant-ph/00080002.
[14] H. Bethe, Z. Phys. **71**, 205 (1931).
[15] L.D. Faddeev and L.A. Takhtajan, J. Sov. Math. **24**, 241 (1984).

[16] V.E. Korepin, N.M. Bogoliubov, and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Function* (Cambridge University Press, Cambridge, 1993).
[17] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 267 (1995).
[18] C.M. Bender and T.T. Wu, Phys. Rev. **184**, 1231 (1969); Phys. Rev. D **7**, 1620 (1973).
[19] C.M. Bender and L.M.A. Bettencourt, Phys. Rev. Lett. **77**, 20 (1996).
[20] W.Y. Keung, E. Kovacs, and U.P. Sukhatme, Phys. Rev. Lett. **60**, 41 (1988).
[21] C. Itzykson and J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[22] C.M. Bender and T.T. Wu, Phys. Rev. Lett. **21**, 406 (1968).
[23] R.F. Casten, N.V. Zamfir, and D.S. Brenner, Phys. Rev. Lett. **71**, 227 (1993).
[24] P.P. Corso, C.L. Lo Cascio, and F. Persico, Phys. Rev. A **58**, 1549 (1998).
[25] M. Carvajal, J.M. Arias, and J. Gomez-Camacho, Phys. Rev. A **59**, 3462 (1999).
[26] S. Sen, R.S. Sinkovits, and S. Chakravarti, Phys. Rev. Lett. **77**, 4855 (1996).
[27] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A **60**, 4171 (1999).
[28] K. Aarset K, A.G. Csaszar, E.L. Sibert, W.D. Allen, H.F. Schaefer, W. Klopper, and J. Noga, J. Chem. Phys. **112**, 4053 (2000).
[29] R.N. Chaudhuri and M. Mondal, Phys. Rev. A **52**, 1850 (1995); **40**, 6080 (1989); **43**, 3241 (1991); R.K. Agrawal and V.S. Varma, *ibid.* **49**, 5089 (1994).
[30] K.C. Ho, Y.T. Liu, C.F. Lo, K.L. Liu, W.M. Kwok, and M.L. Shiu, Phys. Rev. A **53**, 1280 (1996).
[31] I.D. Feranchuk, L.I. Komarov, I.V. Nichipor, and A.P. Ulyanenkov, Ann. Phys. (N.Y.) **238**, 370 (1995).
[32] I.D. Feranchuk and L.I. Komarov, Phys. Lett. **88A**, 211 (1982); K. Yamazaki, J. Phys. A **17**, 345 (1984); H. Mitter and K. Yamazaki, *ibid.* **17**, 1215 (1984).
[33] R. Jauregui and J. Recamier, Phys. Rev. A **46**, 2240 (1992).
[34] H. Scherrer, H. Risken, and T. Leiber, Phys. Rev. A **38**, 3949 (1988).
[35] L. Skala, J. Cizek, J. Dvorak, and V. Spirko, Phys. Rev. A **53**, 2009 (1996); L. Skala, J. Cizek, V. Kapsa, and E.J. Weniger, *ibid.* **56**, 4471 (1997).
[36] I.A. Ivanov, Phys. Rev. A **54**, 81 (1996).
[37] A.S. Dutra, A.S. Castro, and H. Boschi-Filho, Phys. Rev. A **51**, 3480 (1995).
[38] Yu Zhou, J. Mancini, and P.F. Meier, Phys. Rev. A **51**, 3337 (1995).
[39] S.B. Yuste and A.M. Sanchez, Phys. Rev. A **48**, 3478 (1993).
[40] H. Meißner and E.O. Steinborn, Phys. Rev. A **56**, 1189 (1997).
[41] E.J. Weniger, Phys. Rev. Lett. **77**, 2859 (1996); Ann. Phys. (N.Y.) **246**, 133 (1996).
[42] C.R. Handy, Phys. Rev. A **46**, 1663 (1992).
[43] F.A. Desaavedra and E. Buendla, Phys. Rev. A **42**, 5073 (1990).
[44] R.F. Bishop, M.C. Bosca, and M.F. Flynn, Phys. Rev. A **40**, 3484 (1989); R.J. Damburg, R.K. Propin, and Y.I. Ryabykh, *ibid.* **41**, 1218 (1990); R.F. Bishop and M.F. Flynn, *ibid.* **38**, 2211 (1988).
[45] G. Dattoli and A. Torre, Phys. Rev. A **37**, 1571 (1988).
[46] B. Simon, Ann. Phys. (N.Y.) **58**, 76 (1970).